On Equations for the Multi-quark Bound States in Nambu–Jona-Lasinio Model

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In present report we review some preliminary results of investigation of higher orders of mean-field expansion for Nambu–Jona-Lasinio model. We discuss first results of investigation of next-to-next-to-leading order of mean-field expansion equations for four-quark and three-quark Green functions. We have considered equations for Green functions of Nambu–Jona-Lasinio model in mean-field expansion up to third order.

1. INTRODUCTION AND SUMMARY

Multi-particle equations are a traditional basis for a quantum-field description of bound states in particle physics. A well-known example of these equations is the two-particle Bethe-Salpeter equation (BSE) for the two-particle amplitude and for the two-particle bound state [1].The multi-particle (three or more particle) generalizations of the BSE have been also studied. A straightforward generalization of two-particle BSE has been intensively studied in sixties-seventies of last century. A best exposition of these studies can be found in the work of Huang and Weldon [2].

An essential imperfection of the original Bethe-Salpeter approach to multi-particle equations was a full disconnection of the approach with the field-theoretical equations for Green functions (which are known as Schwinger-Dayson equations (SDE)). This imperfection has been eliminated by Dahmen and Jona-Lasinio, which had included the BSE to the field-theoretical Lagrangian formalism with the consideration of functional Legendre transformation with respect to bilocal source of fields [3]. Then this approach has been generalized for multi-particle equations with consideration of Legendre transformation with respect to multi-local sources [4].

However, these theoretical constructions had not solved the principal dynamical problem of quantitative description of real bound states (nucleons, mesons etc.). A solution of BSE-type equations has been founded as a very complicated mathematical tool even for simple dynamical model. There is a main reason of a comparatively small popularity of the method of multi-particle BSE-type equations among the theorists. Much more popular approach to the problem of hadronization in QCD is based on the 't Hooft's conjecture that QCD can be regarded as an effective theory of mesons and glueballs [5]. Subsequently, it was shown by Witten that the baryons could be viewed as the solitons of the meson theory [6]. Futher development of these ideas has been successful and has leaded to the prediction of pentaquark states in baryon spectrum [7].

Nevertheless, the investigations of multi-quark equations are of significant interest due to the much less model assumptions in this approach in comparison with the chiral-soliton models. The solutions of multi-quark equations will provide us almost exhaustive information about the structure of hadrons. There is basic motivation of present work.

We shall investigate Nambu–Jona-Lasinio (NJL) model with quark content which is one of the most successful effective models of QCD in the non-perturbative region (for review see [8], [9]). In overwhelming majority of the investigations, the NJL model has been considered in the mean-field approximation or in the leading order of $1/n_c$- expansion. However, a number of perspective physical applications of NJL model is connected with multi-quark functions (for example: meson decays, pion-pion scattering, baryons, pentaquarks etc.). These multi-quark functions arise in higher orders of mean-field expansion(MFE) for NJL model. To formulate MFE we have used an iteration scheme of solution of SDE with fermion bilocal source [10].

We have considered equations for Green functions of NJL model in MFE up to third order. The leading approximation and next-to-leading order (NLO) of MFE maintain equations for the quark propagator and the two-quark function and also the NLO correction to the quark propagator. The next-to-next-to-leading order (NNLO) of MFE maintains the equations for four-quark and three-quark functions, and next-to-next-to-next-to-leading order(NNNLO) of MFE maintains the equations for six-quark and five-quark functions.

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IPM-LHP06-19May

2. MEAN-FIELD EXPANSION IN BILOCAL-SOURCE FORMALISM. LEADING ORDER AND NLO

We consider NJL model with the Lagrangian

$$\mathcal{L} = \bar{\psi} i\hat{\partial}\psi + \frac{g}{2}\left[(\bar{\psi}\psi)^2 + (\bar{\psi} i\gamma_5\tau^a\psi)^2\right].$$

The Lagrangian is invariant under transformations of chiral group $SU_V(2)\times SU_A(2)$, which correspond to u-d quark sector. A generating functional of Green functions (vacuum expectation values of T-products of fields) can be represented as the functional integral with bilocal source:

$$G(\eta) = \int D(\psi,\bar{\psi}) \exp i\Big\{ \int dx\mathcal{L}$$

$$- \int dxdy\bar{\psi}(y)\eta(y,x)\psi(x)\Big\}.$$

Here $\eta(y,x)$ is the bilocal source of the quark field.

The n-th functional derivative of G over source η is the n-particle (2n-point) Green function:

$$\frac{\delta^n G}{\delta\eta(y_1,x_1)\cdots\delta\eta(y_n,x_n)}\Big|_{\eta=0}$$

$$= i^n < 0T\Big\{\psi(x_1)\bar{\psi}(y_1)\cdots\psi(x_n)\bar{\psi}(y_n)\Big\} \mid 0 >$$

$$\equiv S_n\begin{pmatrix} x_1 & y_1 \\ \cdots & \cdots \\ x_n & y_n \end{pmatrix}.$$

Translational invariance of the functional-integration measure gives us the functional-differential SDE for the generating functional G :

$$\delta(x-y)G + i\hat{\partial}_x\frac{\delta G}{\delta\eta(y,x)} + ig\Big\{\frac{\delta}{\delta\eta(y,x)}tr\left[\frac{\delta G}{\delta\eta(x,x)}\right]$$

$$-\gamma_5\tau^a\frac{\delta}{\delta\eta(y,x)}tr\left[\gamma_5\tau^a\frac{\delta G}{\delta\eta(x,x)}\right]\Big\}$$

$$= \int dx_1\eta(x,x_1)\frac{\delta G}{\delta\eta(y,x_1)}.$$

We shall solve this equation employing the method which proposed in work [11]. A leading approximation is the functional

$$G^{(0)} = \exp\Big\{\mathrm{Tr}\left(S^{(0)} * \eta\right)\Big\}.$$

The leading approximation generates the linear iteration scheme:

$$G = G^{(0)} + G^{(1)} + \cdots + G^{(n)} + \cdots,$$

Functional $G^{(n)}$ is

$$G^{(n)} = P^{(n)}G^{(0)},$$

where $P^{(n)}$ is a polynomial of $2n$ -th order over the bilocal source η .

The unique connected Green function of the leading approximation is the quark propagator. Other connected Green functions appear in the following iteration steps. The quark propagator in the chiral limit is

$$S^{(0)} = (m-\hat{p})^{-1},$$

where m is the dynamical quark mass, which is a solution of gap equation.

A first-order functional is

$$G^{(1)} = \left\{\frac{1}{2}\mathrm{Tr}\left(S_2^{(1)} * \eta^2\right) + \mathrm{Tr}\left(S^{(1)} * \eta\right)\right\}G^{(0)}.$$

The iteration-scheme equations give us the equation for two-particle function $S_2^{(1)}$:

$$S_2^{(1)}\begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = -S_0(x-y')S_0(x'-y)$$

$$+ig\int dx_1\Big\{(S_0(x-x_1)S_0(x_1-y))tr\left[S_2^{(1)}\begin{pmatrix} x_1 & x_1 \\ x' & y' \end{pmatrix}\right]$$

$$-(S_0(x-x_1)\gamma_5\tau^a\tau^{a_1}S_0(x_1-y))tr\left[\gamma_5\tau^a\tau^{a_1}S_2^{(1)}\begin{pmatrix} x_1 & x_1 \\ x' & y' \end{pmatrix}\right]\Big\} \quad (1)$$

and the NLO correction to quark propagator $S^{(1)}$:

$$S^{(1)}(x-y) = ig\int dx_1 S^{(0)}(x-x_1)\Big\{S_2^{(1)}\begin{pmatrix} x_1 & y \\ x_1 & x_1 \end{pmatrix}$$

$$-\gamma_5\tau^a S_2^{(1)}\begin{pmatrix} x_1 & y \\ x_1 & x_1 \end{pmatrix}\gamma_5\tau^a\Big\}$$

$$+ig\int dx_1 S^{(0)}(x-x_1)S^{(0)}(x_1-y)trS^{(1)}(0).$$

The graphical representations of these equations see on Figs. 1 and 2., where the graphical notations of Fig. 3 are used.

These equations can be easily solved, and the solutions contain singlet scalar quark-antiquark bound state with mass $2m$ (sigma-meson) and massless (in

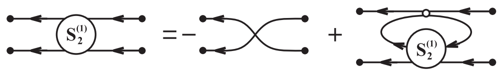

Figure 1: The equation for NLO two-quark function.

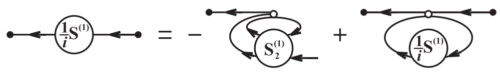

Figure 2: The equation for NLO correction to quark propagator.

the chiral limit) pseudoscalar bound states (pions). To describe the solution of the NLO equation for two-quark function and for future purposes we introduce the composite meson propagators by following way:

a) Let us define scalar-scalar function

$$S_\sigma(x-x') \equiv tr\left[S_2^{(1)}\begin{pmatrix} x & x \\ x' & x' \end{pmatrix}\right] \sim <\bar{\psi}\psi(x)\bar{\psi}\psi(x')> \tag{2}$$

From the equation (1) for two-quark function we obtain (in momentum space)

$$S_\sigma(p) = \frac{1}{ig}(1 - i\Delta_\sigma(p)) \tag{3}$$

Here we define the following function, which we call $\sigma-$meson propagator

$$\Delta_\sigma(p) = \frac{Z(p)}{4m^2 - p^2}, \tag{4}$$

where

$$Z_\sigma(p) = \frac{I_0(4m^2)}{I_0(p^2)}$$

and

$$I_0(p) = \int d\tilde{q}\frac{1}{(m^2-(p+q)^2)(m^2-q^2)}.$$

b) Pseudoscalar-pseudoscalar function is defined as

$$S_\pi^{ab}(x-x') \equiv tr\left[S_2^{(1)}\begin{pmatrix} x & x \\ x' & x' \end{pmatrix}\gamma_5\frac{\tau^a}{2}\gamma_5\frac{\tau^b}{2}\right]$$

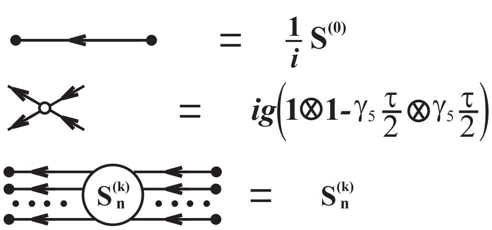

Figure 3: Diagram rules.

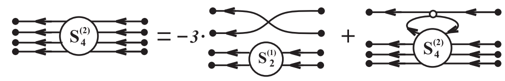

Figure 4: The equation for NNLO four-quark function.

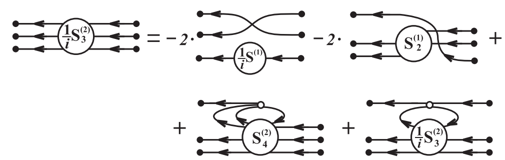

Figure 5: The equation for NNLO three-quark function.

$$\sim <\bar{\psi}\gamma_5\frac{\tau^a}{2}\psi(x)\bar{\psi}\gamma_5\frac{\tau^b}{2}\psi(x')> \tag{5}$$

From the equation (1) for two-quark function we obtain (in momentum space):

$$S_\pi^{ab}(p) = -\frac{1}{ig}(\delta^{ab} - i\Delta_\pi^{ab}(p)). \tag{6}$$

Here we define the pion propagator

$$\Delta_\pi^{ab}(p) = -\frac{\delta^{ab}Z(p)}{p^2}, \tag{7}$$

where $Z_\pi(p) = \frac{I_0(0)}{I_0(p^2)}$.

3. NNLO OF MEAN-FIELD EXPANSION

NNLO(second-order)generating functional is

$$G^{(2)}[\eta] = \left\{\frac{1}{4!}\mathrm{Tr}\left(S_4^{(2)} * \eta^4\right) + \frac{1}{3!}\mathrm{Tr}\left(S_3^{(2)} * \eta^3\right)\right.$$

$$\left.+\frac{1}{2}\mathrm{Tr}\left(S_2^{(2)} * \eta^2\right) + \mathrm{Tr}\left(S^{(2)} * \eta\right)\right\}G^{(0)}.$$

The equations for four-quark and three-quark functions see on Figs. 4 and 5.

The equations for the four-quark function $S_4^{(2)}$ and for the three-quark functions $S_3^{(2)}$ are new, and the equations for two-particle function $S_2^{(2)}$ and propagator $S^{(2)}$ have the same form as the corresponding NLO equation except of the inhomogeneous terms. For NNLO equations these terms contain four-quark function $S_4^{(2)}$ and three-quark $S_3^{(2)}$function.

The equation for the four-quark function has a simple exact solution which is the product of NLO two-quark functions (see Fig. 6). As it seen from this solution, the $\pi\pi-$scattering in NJL model is suppressed, i.e. in the NNLO of MFE this scattering is absent, and it can be arise in the NNNLO (third order) only.

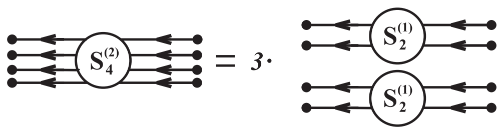

Figure 6: The solution of equation for four-quark function.

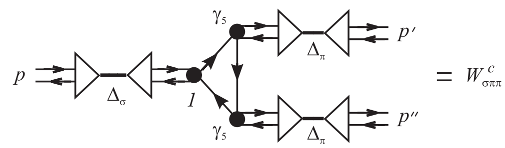

Figure 7: The connected part of sigma-pion-pion-vertex.

4. VERTEX $\sigma\pi\pi$

The existence of above exact solution for the four-quark function gives us a possibility to obtain a closed equation for the three-quark function. As a first step in an investigation of this equation we shall solve a problem of definition of $\sigma\pi\pi-$vertex with composite sigma-meson and pions. Let us introduce a function

$$W^{ab}_{\sigma\pi\pi}(xx'x'') \equiv tr\left[S_3^{(2)}\begin{pmatrix} x & x \\ x' & x' \\ x'' & x'' \end{pmatrix}\gamma_5\frac{\tau^a}{2}\gamma_5\frac{\tau^b}{2}\right]$$

$$\sim < \bar\psi\psi(x)\bar\psi\gamma_5\frac{\tau^a}{2}\psi(x')\bar\psi\gamma_5\frac{\tau^b}{2}\psi(x'') >$$

and define:
a) scalar vertex

$$V_\sigma(xx'x'') \equiv tr\left[S_0(x-x')S_2^{(1)}\begin{pmatrix} x' & x \\ x'' & x'' \end{pmatrix}\right]$$

$$= 2in_c\int dx_1 v_S(xx'x_1)\Delta_\sigma(x_1-x''). \qquad (8)$$

Here: $v_S(xx'x'') = tr_\alpha[S_0(x-x')S_0(x'-x'')S_0(x''-x)]$ is the triangle diagram.
b) pseudoscalar vertex

$$V^{ab}_\pi(xx'x'') \equiv tr\left[S_0(x-x')\gamma_5\frac{\tau^a}{2}S_2^{(1)}\begin{pmatrix} x' & x \\ x'' & x'' \end{pmatrix}\gamma_5\frac{\tau^b}{2}\right]$$

$$= 2in_c\int dx_1 v_P(xx'x_1)\Delta^{ab}_\pi(x_1-x''). \qquad (9)$$

Here: $v_P(xx'x'') = tr_\alpha[S_0(x-x')\gamma_5 S_0(x'-x'')\gamma_5 S_0(x''-x))]$.
With definitions (2)-(9) we obtain for vertex function W^{ab} the following equation:

$$W^{ab}_{\sigma\pi\pi}(xx'x'') = W^{ab}_0(xx'x'')$$

$$+2ign_c\int dx_1 l_S(x-x_1)W^{ab}_{\sigma\pi\pi}(x_1x'x''),$$

where $l_S(x) \equiv tr_\alpha[S_0(x)S_0(-x)]$ is the scalar quark loop. Inhomogeneous term W^{ab}_0 is:

$$W^{ab}_0(xx'x'') = V^{ab}_\pi(xx'x'') + V^{ab}_\pi(xx''x')$$

$$+4ig\int dx_1 V^{a_1a}_\pi(xx_1x')S^{a_1b}_\pi(x_1-x'')$$

$$+4ig\int dx_1 V^{a_1b}_\pi(xx_1x'')S^{a_1a}_\pi(x_1-x')$$

$$-ig\int dx_1(V_\sigma(xx_1x_1) - 4V^{a_1a_1}_\pi(xx_1x_1))S^{ab}_\pi(x'-x'')$$

Using definitions (2)-(9) we have:

$$[W^{ab}_0(xx'x'')]^{con}$$

$$= -2n_c\int dx_1dx_2 v_P(xx_1x_2)[\Delta^{a_1a}_\pi(x_2-x')\Delta^{a_1b}_\pi(x_1-x'')$$

$$+\Delta^{a_1ab}_\pi(x_2-x'')\Delta^{a_1a}_\pi(x_1-x')].$$

The equation for W^{ab} can be easy solved in the momentum space and and a solution is

$$W^{ab}_{\sigma\pi\pi}(pp'p'') = i\Delta_\sigma(p)W^{ab}_0(pp'p'')$$

where p is $\sigma-$mesons momenta, and p', p'' are pion momenta: $p = p' + p''$. The connected part of W^{ab} is an decay amplitude $\sigma \to \pi\pi$. It is has a following form:

$$[W^{ab}_{\sigma\pi\pi}(pp'p'')]^{con}$$

$$= \frac{2n_c}{i}\Delta_\sigma(p)[v_P(pp'p'')+v_P(pp''p')]\Delta^{aa_1}_\pi(p')\Delta^{a_1b}_\pi(p''), \qquad (10)$$

(See also Fig.7.).

5. NNNLO OF MEAN-FIELD EXPANSION

The third-order generating functional is

$$G^{(3)}[\eta] = \left\{\frac{1}{6!}\mathrm{Tr}\left(S_6^{(3)} * \eta^6\right) + \frac{1}{5!}\mathrm{Tr}\left(S_5^{(3)} * \eta^5\right)\right.$$

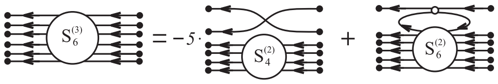

Figure 8: The equation for six-quark function.

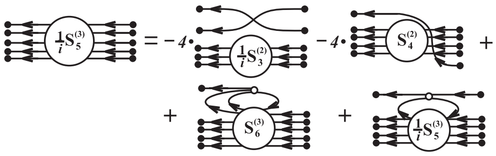

Figure 9: The equation for five-quark function.

$$+\frac{1}{4!}\mathrm{Tr}\Big(S_4^{(3)} * \eta^4\Big) + +\frac{1}{3!}\mathrm{Tr}\Big(S_3^{(3)} * \eta^3\Big)$$

$$+\frac{1}{2}\mathrm{Tr}\Big(S_2^{(3)} * \eta^2\Big) + \mathrm{Tr}\Big(S^{(3)} * \eta\Big)\Big\}G^{(0)}.$$

The graphical representations of equations for six-quark functions and for five-quark function see on Figs. 8 and 9. The equations for the six-quark function and for the five-quark function in our iteration scheme are new, and the equations for four-quark function $S_4^{(3)}$,three-quark function $S_3^{(3)}$, two-quark function $S_2^{(3)}$ and quark propagator $S^{(3)}$ have the same form as the second-order equations except of the inhomogeneous term, which contains the six-quark function and the five-quark function.

Acknowledgments

One of the authors (R.G.J.) would like to express his sincere gratitude to the IPM and Organizing Committee for the invitation and warm hospitality.

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